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On Functions Whose Stationary Points Are Global Minima

by

Israel Zang[1], Eng Ung Choo[2] and Mordecai Avriel[3]

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Abstract

In this paper a characterization of functions whose stationary points are global minima is studied. By considering the level sets of a real function as a point-to-set mapping, and by examining its semi-continuity properties, we obtain a result that a real function, defined on a subset of R^n and satisfying some mild regularity conditions, belongs to the above family if and only if the point-to-set mapping of its level sets is strictly lower semicontinuous. Mathematical programming applications are also mentioned.

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1. Introduction

Point-to-set mappings and their semicontinuity properties have recently been the subject of a number of studies in the mathematical programming literature, see for example Hogan [1], Robinson and Meyer [2] and Zangwill [3]. In Zang and Avriel [4], and Zang, Choo and Avriel [5] it was shown that a necessary and sufficient condition for a real function, defined on a subset of R^n, to be in the family of functions whose local minima are global, is that its level sets, considered as a point-to-set mapping, is lower semicontinuous.

In this work we direct our attention to deriving a similar condition for a function to belong to the family of functions whose <u>stationary points</u> are global minima. This condition will use some stronger semicontinuity properties of level sets.

Let f be a real function on a subset C of R^n and let α be a real number. Consider the level sets of f

(1) $L_f(\alpha) = \{x : x \in C, f(x) \leq \alpha\}$

and the effective domain of $L_f(\alpha)$, i.e. the set

(2) $G_f = \{\alpha : \alpha \in R, L_f(\alpha) \neq \emptyset\}$.

Note that for any real function the set G_f is an interval.

Clearly, $L_f(\alpha)$ is a point-to-set mapping of points in G_f into subsets of R^n.

Recalling the definition of lower semicontinuity of point-to-set mappings we have (see Meyer [6])

<u>Definition 1.</u> <u>The point-to-set mapping</u> $L_f(\alpha)$ <u>is said to be lower semicontinuous (lsc) at a point</u> $\alpha \in G_f$ <u>if</u> $x \in L_f(\alpha)$, $\{\alpha^i\} \subset G_f$,

$\{\alpha^i\} \to \alpha$ imply the existence of a natural number K and a sequence $\{x^i\}$ such that

(3) $\quad x^i \in L_f(\alpha^i)$, $i = K, K+1, \ldots$ and $\{x^i\} \to x$.

If $L_f(\alpha)$ is lsc at every $\alpha \in G_f$ it is said to be lsc on G_f.

Lower semicontinuity of the level set mapping $L_f(\alpha)$ can be used to characterize functions whose local minima are global [4,5]. Level set mappings are monotone in a certain sense: For any two $\alpha \in G_f$, $\bar{\alpha} \in G_f$ such that $\bar{\alpha} < \alpha$ we have $L_f(\bar{\alpha}) \subset L_f(\alpha)$. For such mappings the property to be defined next is stronger than lower semicontinuity.

Definition 2. The point-to-set mapping $L_f(\alpha)$ is said to be strictly lower semicontinuous (slsc) at a point $\alpha \in G_f$ if $x \in L_f(\alpha)$, $\{\alpha^i\} \subset G_f$, $\{\alpha^i\} \to \alpha$ imply the existence of a natural number K, a sequence $\{x^i\}$ and a real number $\beta(x) > 0$ such that

(4) $\quad x^i \in L_f\,[\alpha^i - \beta(x)\|x^i - x\|]$, $i = K, K+1, \ldots$ and $\{x^i\} \to x$

If $L_f(\alpha)$ is slsc at every $\alpha \in G_f$ it is said to be slsc on G_f. Clearly, if $L_f(\alpha)$ is slsc then it is also lsc. Let $B_\delta(x) \subset R^n$ denote an open ball with radius δ centered around x.

Definition 3. A point $\bar{x} \in C$ is a local minimum of f if there exists a $\delta > 0$ such that

(5) $\quad f(x) \geq f(\bar{x})$

for every $x \in C \cap B_\delta(\bar{x})$ and it is a global minimum of f on C if (5) holds for every $x \in C$.

A necessary condition for a local minimum of a real function defined on an open subset of R^n and differentiable at a point $\bar{x}$ is $\nabla f(\bar{x}) = 0$, where ∇f is the gradient of f. A point where the gradient of f vanishes is also called a stationary or critical point of f. Since we shall consider differentiable functions (in a certain sense) which may be defined on a nonopen set it is necessary to extend the classical definition of a stationary point.

We first recall the definition of tangent directions as defined by Hestenes [7].

Definition 4. Let C be a nonempty subset of R^n and let $x^0 \in C$. A vector $y \in R^n$ is called a tangent direction to C at the point x^0 if there exists a sequence $\{x^k\} \subset C$ satisfying $x^k \neq x^0$, $\{x^k\} \to x^0$ and

$$\lim_{k\to\infty} \frac{x^k - x^0}{\| x^k - x^0 \|} = y. \tag{6}$$

The sequence $\{x^k\}$ is said to define the direction y.

If x^0 is an isolated point (x^0 is isolated if $\{x^0\}$ and $C/\{x^0\}$ are disconnected), then there exists no sequence in C converging to x^0, and satisfying $x^k \neq x^0$. In this case $y = 0$ is said to be the only tangent direction to C at x^0. The collection of all tangent directions to C at x^0 will be denoted $Y(C,x^0)$.

We may note here that the "cone of tangents" to a set C at x^0, used in constraint qualification analysis of mathematical programming is the cone consisting of nonnegative multiples of vectors $y \in Y(C,x^0)$, see for example Gould and Tolle [8], or Avriel [9]. Let us define now the concept of y-derivatives.

Definition 5. Let f be a real function on $C \subset R^n$, $\bar{x} \in C$ and $y \in Y(C,\bar{x})$. If there exists a real number $\delta f(\bar{x},y)$ such that

$$(7) \qquad \lim_{k\to\infty} \frac{f(x^k) - f(\bar{x})}{\| x^k - \bar{x} \|} = \delta f(\bar{x},y)$$

for every sequence $\{x^k\} \subset C$ which defines y, then $\delta f(\bar{x},y)$ is called the y-derivative of f at $\bar{x}$.

If C is an open set and $\bar{x} \in C$, then $Y(C,\bar{x}) = R^n$. Suppose f is differentiable at $\bar{x}$. Then $\delta f(\bar{x},y) = y^T \nabla f(\bar{x})$ is the y-derivative of f at $\bar{x}$ for every direction $y \in R^n$.

We now define a stationary point as we shall refer to it in the sequel.

Definition 6. Let f be a real function on $C \subset R^n$ and suppose that f has a y-derivative at $\bar{x} \in C$ for every $y \in Y(C,\bar{x})$. The point $\bar{x}$ is said to be a stationary point of f if $\delta f(\bar{x},y) \geq 0$ for every $y \in Y(C,\bar{x})$.

It was shown by Hestenes [7] that, if $\bar{x}$ is a local minimum of a real function, differentiable at $\bar{x}$, then it is also a stationary point, but not conversely. The concept of stationary points defined here extends the classical definition so that it includes more points which may be possible candidates for a local minimum. Clearly, if $\bar{x}$ lies in the interior of the domain of f, the concept defined here and the classical one, $\nabla f(\bar{x}) = 0$, are identical.

2. Stationary Points and Global Minima

Suppose that f is a real function on $C \subset R^n$. For every real number $\beta > 0$ and $\bar{x} \in C$, let $f_{\beta,\bar{x}}$ be the function defined by $f_{\beta,\bar{x}}(x) = f(x) + \beta\|x - \bar{x}\|$ for $x \in C$.

The following theorem gives two equivalent conditions for the point-to-set mapping $L_f(\alpha)$ to be slsc at a point $\bar{\alpha} \in G_f$.

Theorem 1. *Let* f *be a real function on* $C \subset R^n$ *and let* $\bar{\alpha} \in G_f$. *The following statements are equivalent*:

(a) $L_f(\alpha)$ *is slsc at* $\bar{\alpha}$;

(b) *For every* $\bar{x} \in L_f(\bar{\alpha})$ *such that* $f(\bar{x}) = \bar{\alpha}$, *either* f *has a global minimum at* $\bar{x}$ *or there exists a positive real number* β *such that* $f_{\beta,\bar{x}}$ *does not have a local minimum at* $\bar{x}$;

(c) *For every* $\bar{x} \in L_f(\bar{\alpha})$ *such that* $f(\bar{x}) = \bar{\alpha}$, *either* f *has a global minimum at* $\bar{x}$ *or there exist a sequence* $\{y^i\} \subset C$ *and a positive real number* $\bar{\beta}(\bar{x})$ *such that* $y^i \neq \bar{x}$ *for all* i *and*

$$(8) \qquad y^i \in L_f[\bar{\alpha} - \bar{\beta}(\bar{x})\|y^i - \bar{x}\|] \quad i = 1,2,\ldots \text{ and } \{y^i\} \to \bar{x} .$$

Proof. (a) $\Rightarrow$ (b) . Suppose that $f(x^0) < f(\bar{x}) = \bar{\alpha}$ for some $x^0 \in C$. Then there exists a sequence $\{\alpha^i\} \subset G_f$ such that $\{\alpha^i\} \to \bar{\alpha}$ and $f(x^0) \le \alpha^i < \alpha^{i+1} < \bar{\alpha}$ for all i . By Definition 2, there exist a natural number K , a sequence $\{x^i\} \subset C$ and a real number $\beta > 0$ such that

$$(9) \qquad x^i \in L_f[\alpha^i - \beta\|x^i - \bar{x}\|] \quad i = K,K+1,\ldots \text{ and } \{x^i\} \to \bar{x} .$$

Then,

$$(10) \qquad f_{\beta,\bar{x}}(x^i) = f(x^i) + \beta\|x^i - \bar{x}\| \le \alpha^i < \bar{\alpha} \quad i = K,K+1,\ldots \text{ and } \{x^i\} \to \bar{x} .$$

Thus $\bar{x}$ is not a local minimum of $f_{\beta,\bar{x}}$.

(b) $\Rightarrow$ (c) . Suppose that $f(x^0) < f(\bar{x}) = \bar{\alpha}$ for some $x^0 \in C$. Then there exists a positive real number β such that $\bar{x}$ is not a local minimum of $f_{\beta,\bar{x}}$. Thus there exists a sequence $\{y^i\} \subset C$ such that $f_{\beta,\bar{x}}(y^i) < f_{\beta,\bar{x}}(\bar{x}) = \bar{\alpha}$ and $\{y^i\} \to \bar{x}$. Obviously, $y^i \neq \bar{x}$ for all i , and

$$(11) \qquad f(y^i) < \bar{\alpha} - \beta\|y^i - \bar{x}\| \qquad i = 1,2,\ldots \text{ and } \{y^i\} \to \bar{x} .$$

It is easy to see that (11) implies (8) with $\bar{\beta}(\bar{x}) = \beta$.

(c) $\Rightarrow$ (a) . Let $\bar{x} \in L_f(\bar{\alpha})$, $\{\alpha^i\} \subset G_f$ and $\{\alpha^i\} \to \bar{\alpha}$. If $f(\bar{x}) < \bar{\alpha}$ the definition of strict lower semicontinuity can be easily satisfied at $\bar{x}$. So we need to consider only the case when $f(\bar{x}) = \bar{\alpha}$. If f has a global minimum at $\bar{x}$, then $\alpha^i \geq \bar{\alpha}$ for all i . In this case we let $x^i = \bar{x}$ for all i and β be any positive real number. If $\bar{x}$ is not a global minimum of f , then there exist a sequence $\{y^i\} \subset C$ and a positive real number $\bar{\beta}(\bar{x})$ such that $y^i \neq \bar{x}$ for all i and (8) holds. Let $t^i = \inf\{\alpha^i, \alpha^{i+1}, \ldots\}$ for all i . Then $t^i \leq t^{i+1}$, $t^i \leq \alpha^i$ for all i and $\{t^i\} \to \bar{\alpha}$. For each i , it follows from (8), $y^i \neq \bar{x}$ and $\{t^i\} \to \bar{\alpha}$ that there exists an integer N_i such that

$$(12) \qquad f(y^i) \leq t^j - \tfrac{1}{2}\bar{\beta}(\bar{x})\|y^i - \bar{x}\| \qquad j = N_i+1, N_i+2, \ldots$$

The integers N_i can be chosen so that $i < N_i < N_{i+1}$ for all i . Now, for each $i > N_1$ there exists $i^* = i^*(i)$ such that $N_{i^*}+1 \leq i \leq N_{i^*+1}$. Let $x^i = y^{i^*(i)}$ for all $i > N_1$. Then $\{x^i\} \subset C$ and $\{x^i\} \to \bar{x}$. From (12), we have for each $i > N_1$

(13) $\quad f(x^i) = f(y^{i^*}) \leq t^i - \frac{1}{2}\bar{\beta}(\bar{x})\|y^{i^*} - \bar{x}\| \qquad j = N_{i^*}+1, N_{i^*}+2, \ldots$

In particular, (13) is satisfied for $j = i$, because $i \geq N_{i^*} + 1$. Therefore,

(14) $$f(x^i) \leq t^i - \frac{1}{2}\bar{\beta}(\bar{x})\|x^i - \bar{x}\| \qquad i = N_1+1, N_1+2, \ldots$$

By the definition of t^i and letting $\beta(\bar{x}) = \frac{1}{2}\bar{\beta}(\bar{x})$ we have

(15) $$f(x^i) \leq \alpha^i - \beta(\bar{x})\|x^i - \bar{x}\| \qquad i = N_1+1, N_1+2, \ldots$$

Consequently, $L_f(\alpha)$ is slsc at $\bar{\alpha}$. □

<u>Theorem 2</u>. <u>Let</u> f <u>be a real function on</u> $C \subset R^n$ <u>and let</u> $\bar{x} \in C$ <u>such that</u> $f(\bar{x}) = \bar{\alpha}$. <u>Suppose</u> f <u>has a y-derivative for every</u> $y \in Y(C,\bar{x})$. <u>Then</u> $\bar{x}$ <u>is a stationary point of</u> f <u>if and only if</u> $\bar{x}$ <u>is a local minimum point of</u> $f_{\beta,\bar{x}}$ <u>for every real number</u> $\beta > 0$.

<u>Proof</u>. Suppose $\bar{x}$ is not a local minimum of $f_{\beta,\bar{x}}$ for some real number $\beta > 0$. Then there exists a sequence $\{x^i\} \subset C$ such that $f_{\beta,\bar{x}}(x^i) < f_{\beta,\bar{x}}(\bar{x}) = \bar{\alpha}$ for all i and $\{x^i\} \to \bar{x}$. We may assume that

(16) $$y = \lim_{i \to \infty} \frac{x^i - \bar{x}}{\|x^i - \bar{x}\|}$$

exists. It follows from $f_{\beta,\bar{x}}(x^i) < \bar{\alpha} = f(\bar{x})$ for all i that

$$\frac{f(x^i) - f(\bar{x})}{\|x^i - \bar{x}\|} < -\beta \quad \text{for all } i . \tag{17}$$

Thus $\delta f(\bar{x},y) < 0$ and $\bar{x}$ is not a stationary point of f .

Suppose that $\{x^i\} \subset C$ and $\{x^i\} \to \bar{x}$, $y = \lim_{i \to \infty} \frac{x^i - x}{\|x^i - \bar{x}\|}$ and $\delta f(\bar{x},y) < 0$. Let $\beta = -\frac{1}{2}\delta f(\bar{x},y)$. Then $\beta > 0$ and there exists a natural number N such that

$$\frac{f(x^i) - f(\bar{x})}{\|x^i - \bar{x}\|} < -\beta \qquad i = N, N+1, \ldots \tag{18}$$

Thus

$$f(x^i) + \beta\|x^i - \bar{x}\| < f(\bar{x}) \qquad i = N, N+1, \ldots \quad \text{and} \quad \{x^i\} \to \bar{x} \tag{19}$$

Hence $\bar{x}$ is not a local minimum of $f_{\beta,\bar{x}}$. □

As an immediate result from Theorem 1 and 2 we get

Corollary 1. Let f be a real function on $C \subset R^n$. Suppose that for every $x \in C$, f has a y-derivative at x for each $y \in Y(C,x)$. Then every stationary point of f is a global minimum of f on C if and only if $L_f(\alpha)$ is slsc on G_f .

The following corollary is a special case of Corollary 1.

Corollary 2. Let f be a real differentiable function on an open set containing $C \subset R^n$. Every stationary point of f is a global minimum of f on C if and only if $L_f(\alpha)$ is slsc on G_f .

Let us illustrate now our results by some examples.

Example 1. If x is a stationary point which is not a global minimum, then $L_f(\alpha)$ is not slsc at $\alpha = f(\bar{x})$. In order to show it, consider the function

(20) $$f(x) = (x)^3 .$$

defined on the interval $C = [-1,1]$. This function has a stationary point at $x = 0$, which is not a global minimum. Take any sequence $\{\alpha^i\} \to \bar{\alpha} = 0$ satisfying

(21) $$-1 \leq \alpha^i < 0 \qquad i = 1, 2, \ldots$$

and suppose that $L_f(\alpha)$ is slsc at $\bar{\alpha} = 0$. Then there exist a $\beta(0) > 0$ and a sequence $\{x^i\} \to 0$ satisfying

(22) $$f(x^i) = (x^i)^3 \leq \alpha^i - \beta(0) \|x^i - 0\| \qquad i = 1,2,\ldots .$$

From (21) and (22) we get that $\{x^i\}$ must satisfy

(23) $$x^i < 0 \qquad i = 1,2,\ldots$$

and from (21), (22) and (23) we get

(24) $$(x^i)^3 \leq \alpha^i + \beta(0)x^i < \beta(0)x^i \qquad i = 1,2,\ldots .$$

Dividing by x^i we have

$$(25) \qquad (x^i)^2 > \beta(0) > 0 \qquad i = 1,2,\ldots\ .$$

But $\{x^i\} \to 0$, contradicting (25) . It follows then that $L_f(\alpha)$ is not slsc at $\bar{\alpha} = 0$. ‖

Example 2. Let f be the function defined on $[-2\Pi,1]$ as follows:

$$(26) \qquad f(x) = \begin{cases} \dfrac{2x + 5\Pi}{\Pi} & \text{if } -2\Pi \le x \le -\Pi \\ \sin(x - \frac{\Pi}{2}) + 2 & \text{if } -\Pi < x < 0 \\ e^{2x} & \text{if } 0 \le x \le 1\ . \end{cases}$$

This function is illustrated in the following figure

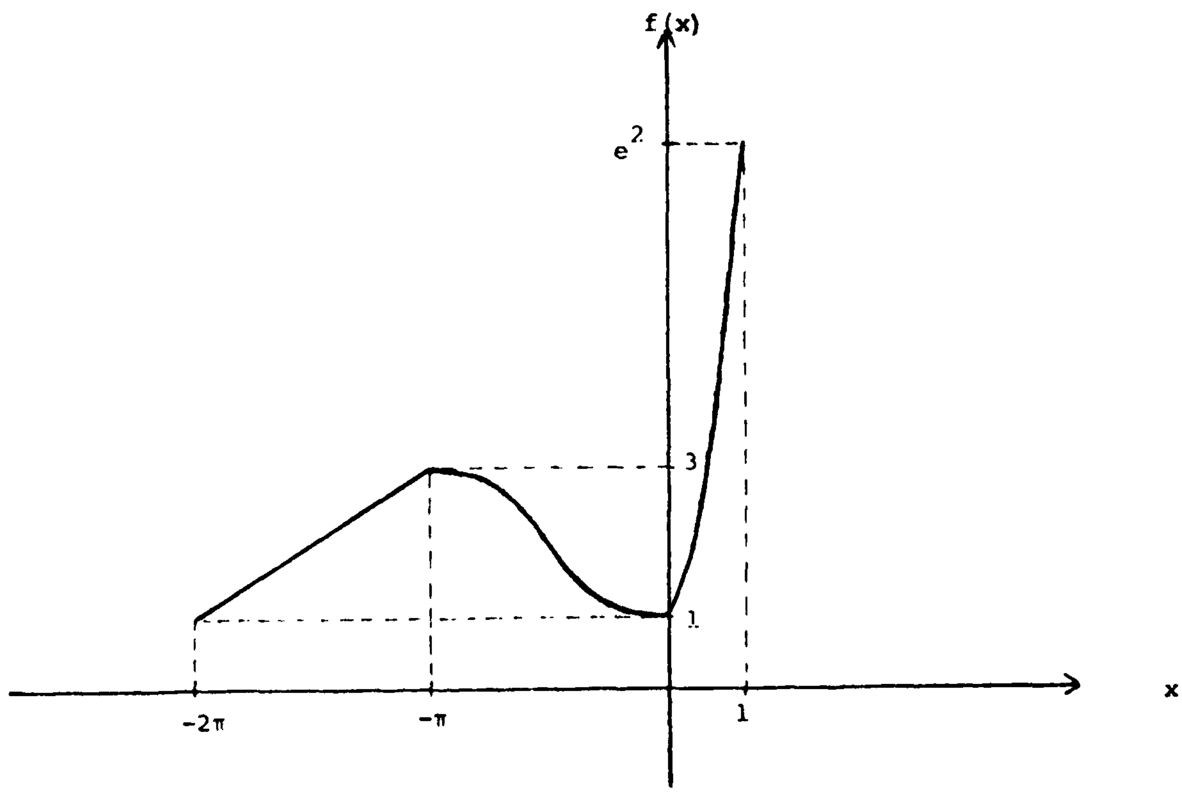

Figure 1

The points -2Π and 0 are the only stationary points of f on $[-2\Pi,1]$ and we see that they are also global minima of f on $[-2\Pi,1]$. The function f has a $(+1)$-derivative $= \frac{2}{\Pi}$ at $x = -2\Pi$ and a (-1)-derivative $= -2e^2$ at $x = 1$. It has both $(+1)$ and (-1) derivatives at each $x \in (-2\Pi,1)$. Hence at every $x \in [-2\Pi,1]$, f has a y-derivative for each $y \in Y([-2\Pi,1],x)$. It follows from Corollary 1 that L_f is slsc on $G_f = [1,\infty)$. Note that even though f has local maxima at $-\Pi$ and 1, they are not stationary points of f. ‖

Let us look now at the family of pseudoconvex functions, defined by Mangasarian [10]: A real differentiable function f on an open set containing the convex set $C \in R^n$ is said to be pseudoconvex on C, if $\tilde{x} \in C$, $\bar{x} \in C$ and

(27) $$f(\tilde{x}) < f(\bar{x})$$

implies

(28) $$(\tilde{x} - \bar{x})^T \nabla f(\bar{x}) < 0 .$$

It is well known that for these functions every point $\bar{x}$ satisfying $\nabla f(\bar{x}) = 0$ is a global minimum. The proof of the following theorem is omitted.

Theorem 3. Let f be a real differentiable function on an open set containing a convex set $C \subset R^n$. If f is pseudoconvex on C, then $L_f(\alpha)$ is slsc on G_f.

The converse result, of course, does not hold since there are functions whose stationary points are global minima but they are not pseudoconvex.

3. An Application to Mathematical Programming

Let us present now some sufficient conditions for a global minimum in a general mathematical program given by

(29) (P) $$\min f(x)$$

subject to

(30) $$g_i(x) \geq 0 \qquad i = 1,\ldots,m$$

(31) $$h_j(x) = 0 \qquad j = 1,\ldots,p\ .$$

We state the next theorem without proof.

Theorem 4. Let $f, g_1 \ldots, g_m$, $h_1 \ldots, h_p$ be real functions on the set

(32) $$X = \{x : x \in R^n,\ g_i(x) \geq 0,\ i=1,\ldots,m,\ h_j(x) = 0,\ j=1,\ldots,p\}\ .$$

Suppose that every stationary point of f in X is a global minimum of f on X. If there exist $x^* \in R^n$, $\lambda^* \in R^m$, $\mu^* \in R^p$ satisfying

(33) $$\delta f(x^*,y) - \sum_{i=1}^{m} \lambda_i^* \delta g_i(x^*,y) - \sum_{j=1}^{p} \mu_j^* \delta h_j(x^*,y) \geq 0$$

for every $y \in Y(X,x^*)$ and

(34) $$\lambda_i^* g_i(x^*) = 0 \qquad i = 1,\ldots,m$$

(35) $$g_i(x^*) \geq 0 \qquad i = 1,\dots,m$$

(36) $$h_j(x^*) = 0 \qquad j = 1,\dots,p$$

(37) $$\lambda^* \geq 0 \ ,$$

<u>then</u> x^* <u>is a global optimum of problem (P)</u>.

Note that these sufficient conditions do not assume convexity or its conventional generalizations (e.g. quasiconvexity, pseudoconvexity) of the functions involved. If f, $g_1,\dots,g_m$, $h_1,\dots,h_p$ are differentiable on an open set containing X then (33) becomes

(38) $$y^T\left(\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) - \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*)\right) \geq 0 \ .$$

4. <u>Some Properties of Functions Having Slsc Level Sets</u>.

We shall consider from now on functions defined on subsets of R^n in the extended sense, i.e. f is a real function on $C \subset R^n$ means that $f(x) = +\infty$ if $x \notin C$.

Let f_1 and f_2 be real functions on the subsets C_1 and C_2 of R^n respectively. The <u>infimal convolution</u> of these two functions is defined as

(39) $$g(x) = (f_1 \,\square\, f_2)(x) = \inf\{f_1(x^1) + f_2(x^2) : x^1 \in C_1, x^2 \in C_2, x^1 + x^2 = x\}$$

and it is a real function on the set $\bar{C} = C_1 + C_2 = \{x^1 + x^2 : x^1 \in C_1, x^2 \in C_2\}$.

Let f be a real function on $C \subset R^n$. Then the nonnegative right scalar multiplication is defined as

$$(40) \qquad h(x) = f\lambda(x) = \begin{cases} \lambda f(x/\lambda) & \text{for } \lambda > 0 \\ 0 & \text{for } \lambda = 0,\ x = 0 \\ +\infty & \text{otherwise .} \end{cases}$$

These two operations are defined in Rockafellar [11]. It is noted there that the infimal convolution corresponds to summation of the epigraphs of the functions f_1 and f_2. If it is possible to replace the infimum in (39) by minimum, we have

$$(41) \qquad \mathrm{epi}(g) = \mathrm{epi}(f_1) + \mathrm{epi}(f_2) \ .$$

It is also noted in [11] that nonnegative right scalar multiplication corresponds to nonnegative scalar multiplication of the epigraph of f and

$$(42) \qquad \mathrm{epi}(h) = \lambda\ \mathrm{epi}(f) \quad \text{for} \quad \lambda > 0 \ .$$

The family of convex functions is closed under these two operations. Let us show now that the family of real functions, defined on compact sets and having slsc level sets, is also closed under these two operations (a real function f on $C \subset R^n$ is said to be closed if its epigraph is closed relative to $G \times R$).

Theorem 5. Let f_1 and f_2 be real closed functions on the compact subsets C_1 and C_2 of R^n, having slsc levels sets on G_{f_1} and G_{f_2} respectively.

Then $L_g(\alpha)$, *where* g *is defined by (39), is slsc on the set*

$$G_g = \{\alpha \in R : L_g(\alpha) \neq \emptyset\} = G_{f_1} + G_{f_2} . \tag{43}$$

Proof. By (39), the closedness of f_1 , f_2 and the compactness of C_1 , C_2 we have

$$g(x) = (f_1 \square f_2)(x) = \min\{f_1(x^1) + f_2(x^2) : x^1 \in C_1 , x^2 \in C_2, x^1 + x^2 = x\} . \tag{44}$$

Let $\bar{\alpha} \in G_g$. If $\bar{\alpha}$ is the global minimum value of g on $C_1 + C_2$, then $L_g(\alpha)$ is slsc at $\bar{\alpha}$. Suppose $\bar{\alpha}$ is not the global minimum value. If $g(\bar{x}) = \bar{\alpha}$, there exist $\tilde{x} \in C_1$ and $\hat{x} \in C_2$ such that

$$\tilde{x} + \hat{x} = \bar{x} \text{ and } f_1(\tilde{x}) + f_2(\hat{x}) = g(\bar{x}) = \bar{\alpha} . \tag{45}$$

Since $L_{f_1}(\alpha)$ and $L_{f_2}(\alpha)$ are slsc on G_{f_1} and G_{f_2} respectively, it follows from Theorem 1 that there exist real numbers $\beta(\tilde{x}) > 0$, $\beta(\hat{x}) > 0$ and sequences $\{\tilde{x}^i\} \subset C_1$, $\{\hat{x}^i\} \subset C_2$, satisfying $\tilde{x}^i \neq \tilde{x}$, $\hat{x}^i \neq \hat{x}$ for all i , such that

$$\tilde{x}^i \in L_{f_1}\left(f_1(\tilde{x}) - \beta(\tilde{x})\|\tilde{x}^i - \tilde{x}\|\right) \quad i=1,2,\ldots \text{ and } \{\tilde{x}^i\} \to \tilde{x} \tag{46}$$

and

$$\hat{x}^i \in L_{f_2}\left(f_2(\hat{x}) - \beta(\hat{x})\|\hat{x}^i - \hat{x}\|\right) \quad i=1,2,\ldots \text{ and } \{\hat{x}^i\} \to \hat{x} . \tag{47}$$

By (45), (46) and (47), we get

$$(48) \quad f_1(\tilde{x}^i) + f_2(\hat{x}^i) \leq g(\bar{x}) - \beta(\tilde{x})\|\tilde{x}^i - \tilde{x}\| - \beta(\hat{x})\|\hat{x}^i - \hat{x}\| \quad i = 1,2,\ldots$$

and $\{\tilde{x}^i + \hat{x}^i\} \to \bar{x}$.

Let $\beta(\bar{x}) = \min\{\beta(\tilde{x}), \beta(\hat{x})\}$. It follows from (48) that

$$(49) \quad f_1(\tilde{x}^i) + f(\hat{x}^i) \leq g(\bar{x}) - \beta(\bar{x})\left(\|\tilde{x}^i - \tilde{x}\| + \|\hat{x}^i - \hat{x}\|\right) \quad i = 1,2,\ldots$$

$$(50) \quad \leq g(\bar{x}) - \beta(\bar{x})\|(\tilde{x}^i + \hat{x}^i) - \bar{x}\| \quad i = 1,2,\ldots$$

and $\{\tilde{x}^i + \hat{x}^i\} \to \bar{x}$

By taking $y^i = \tilde{x}^i + \hat{x}^i$ for every i , we have $y^i \neq \bar{x}$ for all i ,

$$(51) \quad y^i \in L_g\left(\bar{\alpha} - \beta(\bar{x})\|y^i - \bar{x}\|\right) \quad i = 1,2,\ldots \quad \text{and} \quad \{y^i\} \to \bar{x} .$$

By Theorem 1, $L_g(\alpha)$ is slsc at $\bar{\alpha}$. Thus $L_g(\alpha)$ is slsc on G_g . □

<u>Theorem 6.</u> <u>Let</u> f <u>be a real function on</u> $C \subset R^n$ <u>and suppose that</u> $L_f(\alpha)$ <u>is slsc on</u> G_f . <u>Also let</u> h <u>be defined by (40). Then for every</u> $\lambda \geq 0$, $L_h(\alpha)$ <u>is slsc on the set</u> G_h <u>given by</u>

$$(52) \quad G_h = \begin{cases} \lambda G_f & \text{if } \lambda > 0 \\ \{\alpha \in R : \alpha \geq 0\} & \text{if } \lambda = 0 . \end{cases}$$

Proof. The result is obvious when $\lambda = 0$. Suppose $\lambda > 0$. Let $\bar{\alpha} \in G_h$. If $\bar{\alpha}$ is the global minimum value of h on λC, then $L_h(\alpha)$ is slsc at $\bar{\alpha}$. Suppose $\bar{\alpha}$ is not the global minimum value and $h(\bar{x}) = \lambda f(\bar{x}/\lambda) = \bar{\alpha}$. Then $f(\bar{x}/\lambda) = \bar{\alpha}/\lambda$. Since $\bar{\alpha}$ is not the global minimum value of h on λC, hence $\bar{\alpha}/\lambda$ is not the global minimum value of f on C. Since L_f is slsc on G_f, by Theorem 1 there exist a positive real number $\beta(\bar{x})$ and a sequence $\{x^i\} \subset C$ satisfying $x^i \neq \bar{x}/\lambda$,

$$(53) \qquad x^i \in L_f[(\bar{\alpha}/\lambda) - \beta(\bar{x})\|x^i - (\bar{x}/\lambda)\|] \quad i = 1,2,\ldots \quad \text{and} \quad \{x^i\} \to \bar{x}/\lambda$$

Let $y^i = \lambda x^i$ for all i. Then $y^i \neq \bar{x}$ for all i. It follows from (40) and (53) that

$$(54) \qquad y^i \in L_h[\bar{\alpha} - \beta(\bar{x})\|y^i - \bar{x}\|] \quad i = 1,2,\ldots \quad \text{and} \quad \{y^i\} \to \bar{x}$$

By Theorem 1, $L_h(\alpha)$ is slsc at $\bar{\alpha}$. Consequently, $L_h(\bar{\alpha})$ is slsc on G_h.

Q.E.D.

As a result from the last two theorems we conclude that the family of closed functions, defined on compact sets and having the property that every stationary point is a global minimum, is closed under infimal convolution and nonnegative right scalar multiplication.

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In this paper a characterization of functions whose stationary points are global minima is studied. By considering the level sets of a real function as a point-to-set mapping, and by examining its semi-continuity properties, we obtain a result that a real function, defined on a subset of R^n and satisfying some mild regularity conditions, belongs to the above family if and only if the point-to-set mapping of its level sets is strictly lower semicontinuous. Mathematical programming applications are also mentioned.

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